

Relaxation Times

In order to appreciate the parameters of the Galactic disk, it is necessary to understand the concept of “relaxation time”. How long will it take for a star to gravitationally scatter enough so that it loses information about its origin?

One way of parameterizing this is through energy exchange: when does the kinetic energy exchanged during stellar encounters equal the star’s original kinetic energy? In other words,

$$T_E \Rightarrow \sum (\Delta E)^2 \approx E$$

Alternatively, one can define the relaxation time as the time it takes a star to lose all memory of its original trajectory. In this case

$$T_D \Rightarrow \sum \sin^2 \varphi \approx 1$$

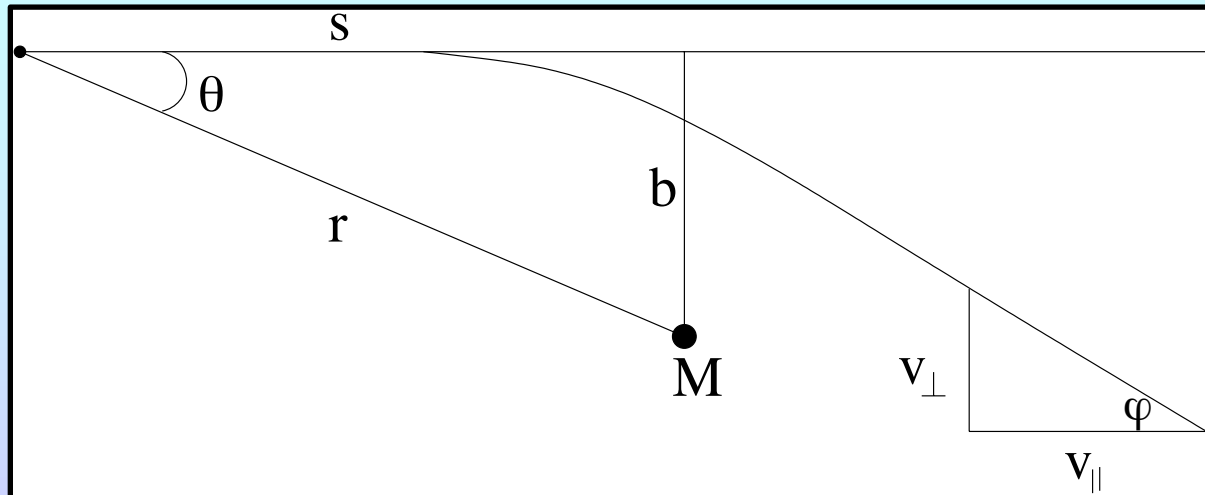
The two values are closely related. Since it’s easier to derive the latter quantity, we’ll use it as the definition of relaxation time.

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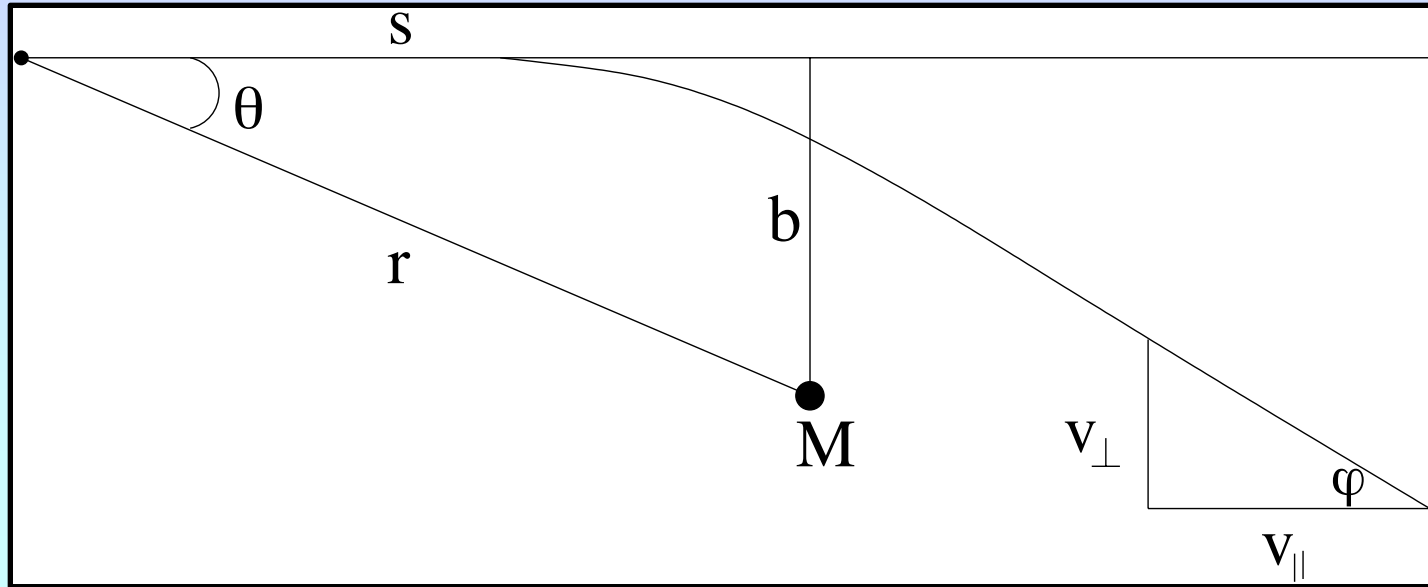
Assume that

- all deflections are two-body encounters
- each encounter is statistically independent of all previous encounters
- close encounters are insignificant compared to long-range encounters, so that during each encounter, $|\Delta E| \ll E$.

Under these assumptions, all the deflections are small ($\sin \varphi \ll 1$), so we can use the Born approximation, in which $v_{\text{init}} \sim v_{\text{final}} \sim v$. Now let's consider a star's gravitational encounter with another object.



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For a single encounter, the deflection angle, φ , can be computed as a function of the initial impact parameter, b , by

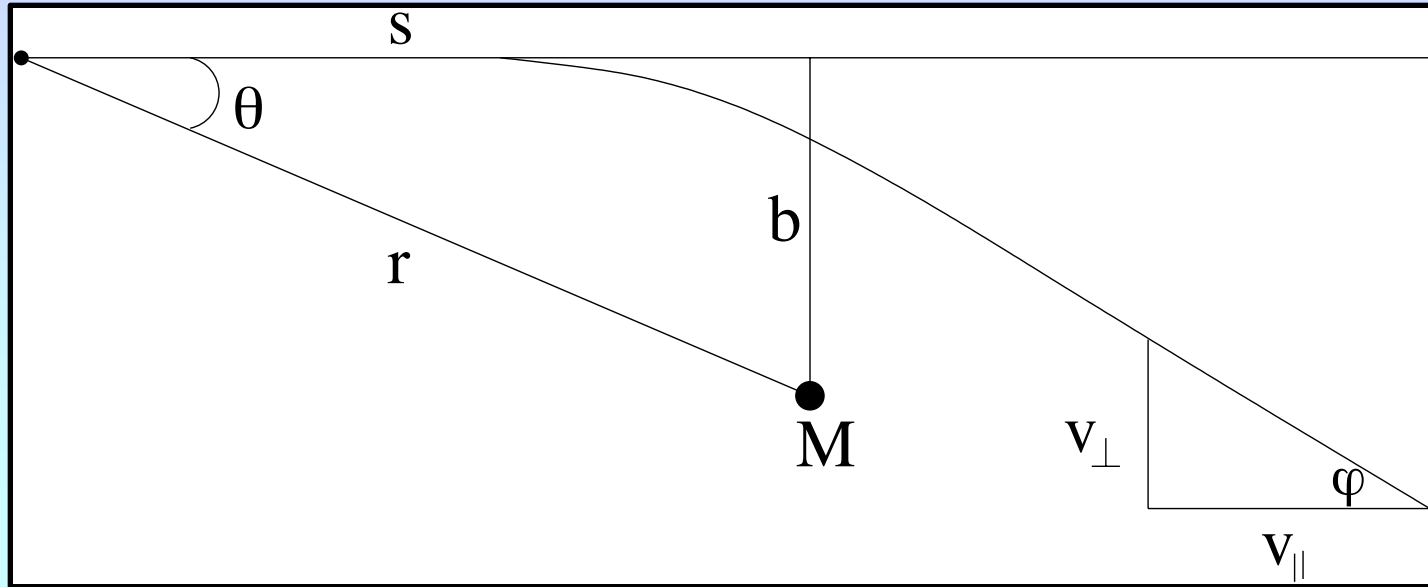
$$F_{\perp} = m \frac{dv_{\perp}}{dt} \Rightarrow v_{\perp} = \int_{-\infty}^{\infty} dv_{\perp} = \frac{1}{m} \int_{-\infty}^{\infty} F_{\perp} dt$$

From the geometry of the encounter

$$F_{\perp} = F \sin \theta = F \left(\frac{b}{r} \right) = \left(\frac{GMm}{r^2} \right) \left(\frac{b}{r} \right)$$

and from the Born approximation, $v_{\parallel} dt = v dt = ds \rightarrow dt = ds / v$, so

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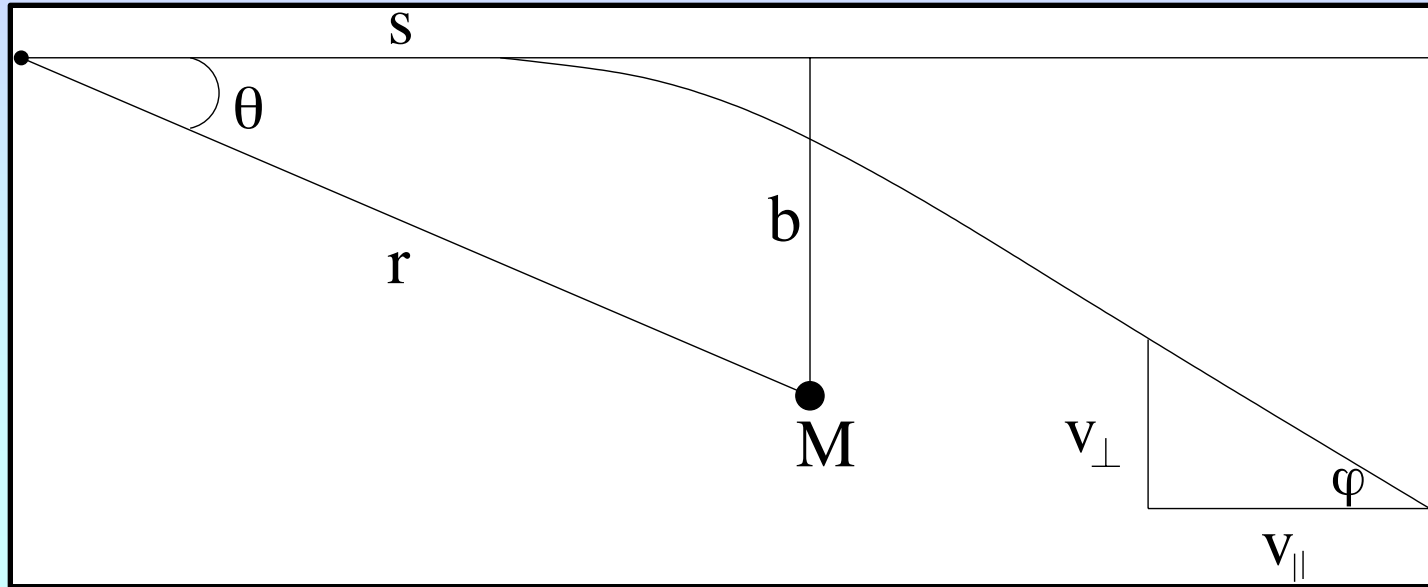
$$v_{\perp} = \frac{1}{m} \int_{-\infty}^{\infty} F_{\perp} dt = \frac{2}{m} \int_0^{\infty} \left(\frac{GMm}{r^2} \right) \left(\frac{b}{r} \right) \left(\frac{1}{v} \right) ds$$

Since $r = (s^2 + b^2)^{1/2}$

$$v_{\perp} = \frac{2GM}{v} \int_0^{\infty} \frac{b}{(s^2 + b^2)^{3/2}} ds = \frac{2GM}{v} \int_0^{\infty} \frac{ds/b}{(1 + (s/b)^2)^{3/2}}$$

or, letting $x = s/b$,

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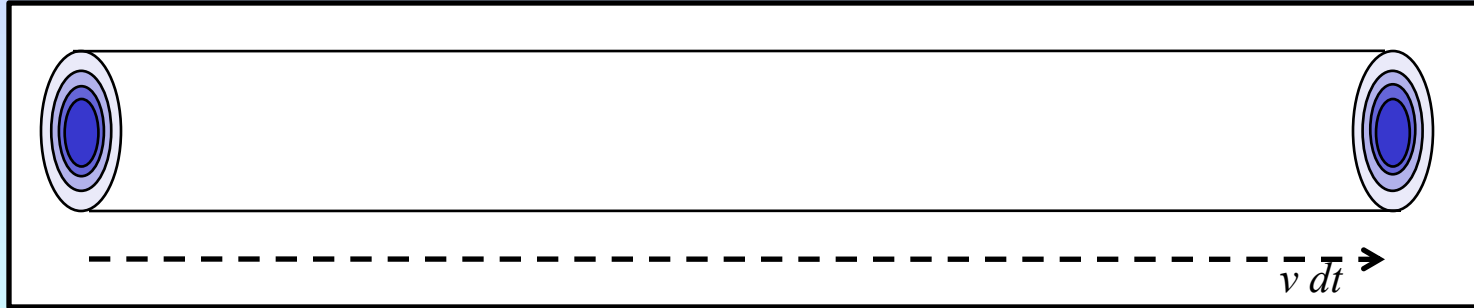


$$v_{\perp} = \frac{2GM}{vb} \int_0^{\infty} \frac{dx}{(1+x^2)^{3/2}} = \frac{2GM}{vb} \cdot \frac{x}{(1+x^2)^{1/2}} \bigg|_0^{\infty} = \frac{2GM}{vb}$$

for small deflections, $\tan \varphi \approx \varphi \approx v_{\perp}/v$. So for the case where all the particles have the same mass ($m = M$), the deflection angle as a function of impact parameter, b , is

$$\varphi(b) = \frac{2GM}{v^2 b} = \frac{2Gm}{v^2 b}$$

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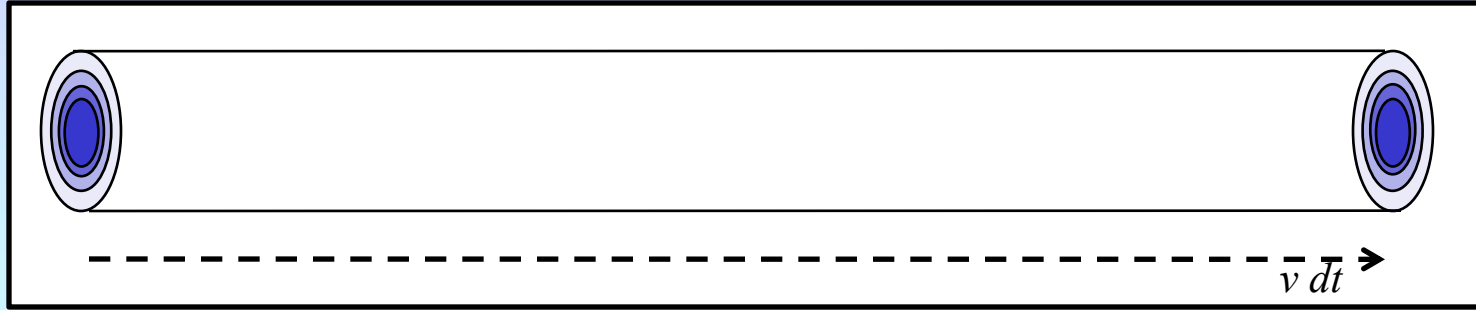
Now, let's sum over all possible collisions. The number of collisions per time dt depends on the impact parameter, the distance a star travels in dt , and the density of stars in the stellar system, n , i.e.,

$$dN_{\text{coll}} = (2\pi b db) \cdot (v dt) \cdot n$$

Consequently,

$$\begin{aligned} \sum \sin^2 \varphi &\approx \sum \varphi^2 = 1 = \int_0^{T_D} \int_{b_{\min}}^{b_{\max}} (2\pi b db) (v dt) n \varphi^2(b) \\ &= \int_0^{T_D} \int_{b_{\min}}^{b_{\max}} (2\pi b db) (v dt) n \left(\frac{2Gm}{v^2 b} \right)^2 \\ &= \frac{8\pi G^2 m^2 n}{v^3} T_D \int_{b_{\min}}^{b_{\max}} \frac{db}{b} = \frac{8\pi G^2 m^2 n}{v^3} T_D \ln \left(\frac{b_{\max}}{b_{\min}} \right) \end{aligned}$$

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The only parameter still needed is b_{\max}/b_{\min} , and since this only enters in as the \ln , the exact numbers chosen aren't very important. One can start with the fact that no deflection angle can be larger than π , so

$$\varphi = \frac{2Gm}{v^2 b_{\min}} = \pi \quad \Rightarrow \quad b_{\min} = \frac{2Gm}{\pi v^2}$$

Conversely, no impact parameter can be greater than the mean stellar distance, i.e.,

$$n = \frac{1}{(4/3)\pi b_{\max}^3} \quad \Rightarrow \quad b_{\max} = \left(\frac{3}{4\pi n} \right)^{1/3}$$

So

$$\left(\frac{8\pi G^2 m^2 n}{v^3} \right) T_D \ln \left\{ \frac{b_{\max} \pi v^2}{2Gm} \right\} = 1$$

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Our simple dynamical relaxation time is therefore

$$T_D = \frac{v^3}{8\pi G^2 m^2 n} \bigg/ \ln \left\{ \frac{b_{\max} v^2 \pi}{2Gm} \right\} = 2.1 \times 10^9 \left(\frac{v^3}{m^2 n} \right) \bigg/ \ln \left\{ 365 \left(\frac{b_{\max} v^2}{m} \right) \right\} \text{ yr}$$

with v in km/s, M in M_\odot , and n in stars/pc³. A more rigorous derivation by Chandrasekhar gives

$$T_D = \frac{v^3}{8\pi G^2 m^2 n H(\chi) \ln(b_{\max} v^2 / 2GM)} \text{ and } T_E = \frac{v^3}{32\pi G^2 m^2 n G(\chi) \ln(b_{\max} v^2 / 2GM)}$$

where $H(\chi)$ and $G(\chi)$ are factors of order unity that depend on the stellar phase-space distribution function. Finally, Ostriker & Davidson (1968) give an improved recursive expression for relaxation time:

$$T_P = \frac{v^3}{8\pi G^2 m^2 n} \bigg/ \ln \left\{ \frac{v^3 T_P}{2Gm} \right\}$$

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If the expression for relaxation time seems difficult to remember, we can simplify things. First, let's define the crossing time of a system with radius, R , as $t_{\text{cross}} \sim 2 R / v$. The ratio of relaxation time to crossing time is then

$$\frac{T_D}{t_{\text{cross}}} = \frac{v^3}{8\pi G^2 m^2 n \ln(b_{\text{max}} / b_{\text{min}})} \bigg/ \frac{2R}{v} = \frac{v^4}{16\pi G^2 m^2 n R \ln(b_{\text{max}} / b_{\text{min}})}$$

Now let's assume that the system is in virial equilibrium. In that case

$$v^2 \approx \frac{GM}{R} \approx \frac{GNm}{R}$$

where M is the total mass of the cluster and N the total number of stars. If we substitute for velocity, then

$$\frac{T_D}{t_{\text{cross}}} = \frac{G^2 N^2 m^2}{16\pi G^2 m^2 n R^3 \ln(b_{\text{max}} / b_{\text{min}})} = \frac{N^2}{16\pi n R^3 \ln(b_{\text{max}} / b_{\text{min}})}$$

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But note that the total number of stars (N) is related to the stellar density (n) by

$$N = \frac{4}{3}\pi R^3 \cdot n$$

So

$$\frac{T_D}{t_{\text{cross}}} = \frac{N^2}{16\pi R^3} \left(\frac{4\pi R^3}{3N} \right) / \ln \left(\frac{b_{\text{max}}}{b_{\text{min}}} \right) = \frac{N}{12 \ln(b_{\text{max}} / b_{\text{min}})}$$

Finally, if we again assume virial equilibrium

$$b_{\text{max}} = \left(\frac{3}{4\pi n} \right)^{1/3} \propto \frac{R}{N^{1/3}}; \quad b_{\text{min}} = \frac{2Gm}{\pi v^2} \propto \frac{R}{N} \quad \Rightarrow \quad \frac{b_{\text{max}}}{b_{\text{min}}} \propto N^{2/3}$$

So, to within a factor of order unity, the relaxation only depends on the total number of stars in the system and their crossing time, i.e.,

$$T_D = \frac{N}{8 \ln(N)} t_{\text{cross}}$$

Relaxation Times

Note: The previous analysis does not just apply to stars in a galaxy or cluster. The exact same proof applies to

- The thermalization of particles in the interstellar medium: How long until an electron (or a proton) ejected from some object with velocity v becomes part of the Maxwellian distribution of electron velocities.
- The characterization of X-ray clusters: is the X-ray gas surrounding galaxy clusters in thermal equilibrium (i.e., isothermal)?
- Dynamical friction: when two galaxies collide, how long until mass segregation occurs?
- Gravitational lenses: how does the angle between images relate to the mass of the lens and the distances to the object and lens.

(and several other uses)

Star Clusters

In the solar neighborhood, the Sun is moving ~ 20 km/s with respect to the surrounding stars. A density of ~ 1 star pc^{-3} then implies a relaxation time of $\sim 10^{14}$ yr. The Sun's orbit about the center of the Galaxy is not gravitationally affected by other stars.

On the other hand, giant molecular clouds have masses that are $\sim 10^8 M_{\odot}$. Although the number density of clouds is lower, it's not 10^{16} times lower! The masses of these clouds are therefore high enough to scatter stars out of their circular orbits, to produce σ_R , σ_{θ} , and σ_z . The result is an increase in scale height with population age.

Note that because all the objects are (approximately) in a plane, one would expect $\sigma_R > \sigma_z$. This is what is seen.

Star Clusters

The Milky Way currently has two types of star clusters:

- Open clusters: young systems containing $\sim 10^3$ stars
- Globular clusters: very old systems containing $> 10^5$ stars

Our Milky Way no longer seems to make very massive star clusters.

M67 open cluster



The Π and χ Persei open clusters

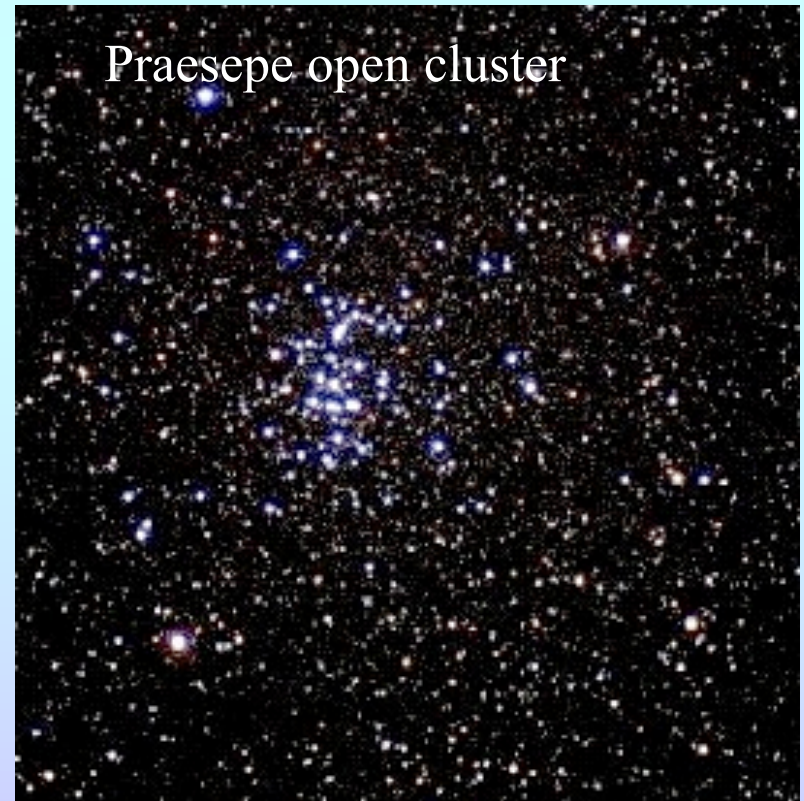


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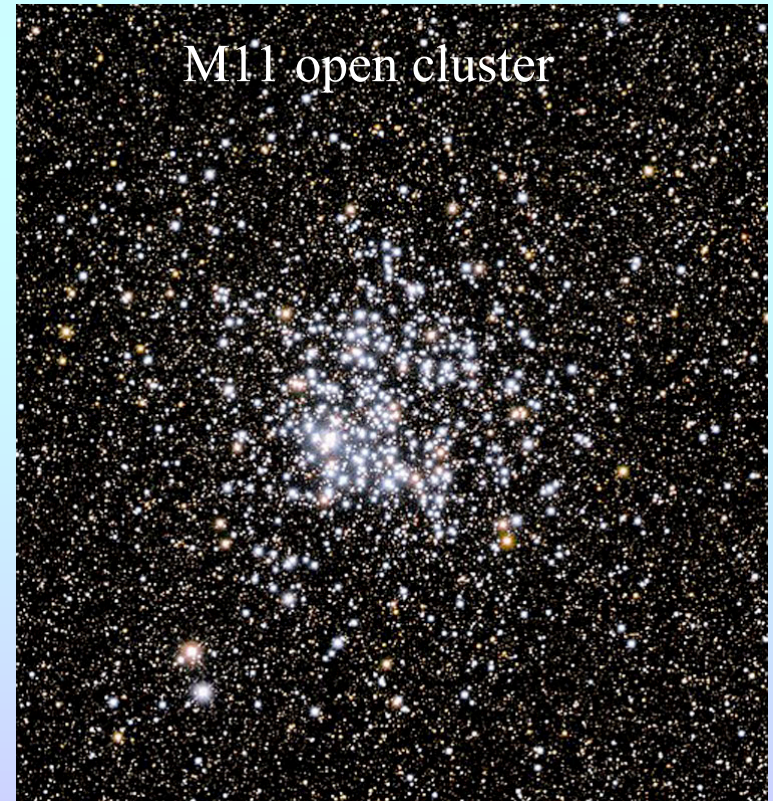
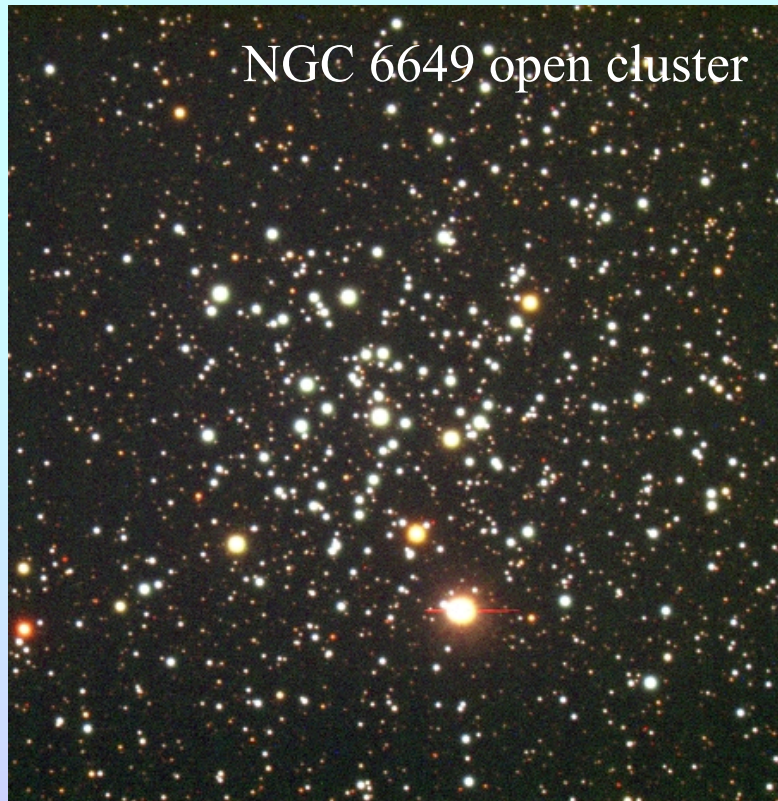


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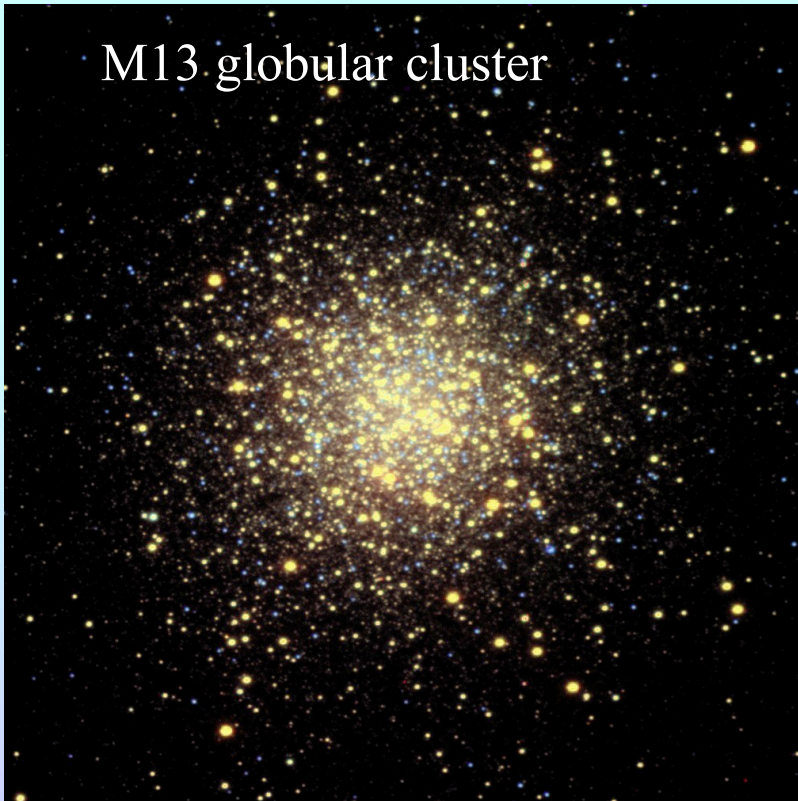
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M13 globular cluster

M3 globular cluster



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Open clusters have typical half-light diameters of ~ 5 pc and velocity dispersions of ~ 5 km/s; globular clusters have half-light diameters of ~ 20 pc and $\sigma \sim 20$ km/s. These numbers imply relaxation times of < 1 Gyr, hence stellar encounters are non-negligible. The stars are exchanging energy!

Isothermal Spheres

In globular clusters, where stars have had plenty of time to exchange energy, the stars approach an equipartition state, where

$$N(E)dE = N_0 \left\{ \frac{4E}{\pi (kT)^3} \right\}^{\frac{1}{2}} e^{-(E+\Omega(r))/kT} dE \quad \text{where} \quad E = \frac{1}{2}m\langle v^2 \rangle = \frac{3}{2}kT$$

In other words, the stars distribute their velocities in a Maxwellian fashion, with a different characteristic velocity at every position in the cluster's potential, $\Omega(r)$. Thus

$$N(v)dv = N_0 \left(\frac{m}{2\pi kT} \right)^{3/2} 4\pi v^2 \exp \left\{ -\frac{mv^2}{2kT} - \frac{\Omega(r)}{kT} \right\} dv$$

or, more clearly,

$$N(v)dv = N(r) \left(\frac{m}{2\pi kT} \right)^{3/2} 4\pi v^2 \exp \left\{ -\frac{mv^2}{2kT} \right\} dv \quad \text{and} \quad N(r) = N_0 \exp \left\{ -\frac{\Omega(r)}{kT} \right\}$$

Isothermal Spheres

Now let's make a simple cluster where all the stars have the same mass. In that case

$$\rho(r) = N(r) \cdot m = \rho_0 e^{-\Omega(r)/kT} \quad \Rightarrow \quad \ln \rho(r) = \ln \rho_0 - \frac{\Omega(r)}{kT} \quad \Rightarrow \quad \frac{d\Omega}{dr} = -kT \frac{d \ln \rho}{dr}$$

If we throw this into the spherically symmetric Poisson equation, we obtain an equation for the structure of this “isothermal” cluster.

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Omega}{dr} \right) = 4\pi G \rho \quad \Rightarrow \quad \frac{d}{dr} \left(r^2 \frac{d \ln \rho}{dr} \right) = -\frac{4\pi G}{kT} r^2 \rho$$

One solution to this equation is a simple inverse square law. If we substitute velocity dispersion for temperature using $\langle v^2 \rangle = 3kT/m$, and define $\sigma^2 = \langle v^2 \rangle/3$ as the line-of-sight velocity dispersion, then

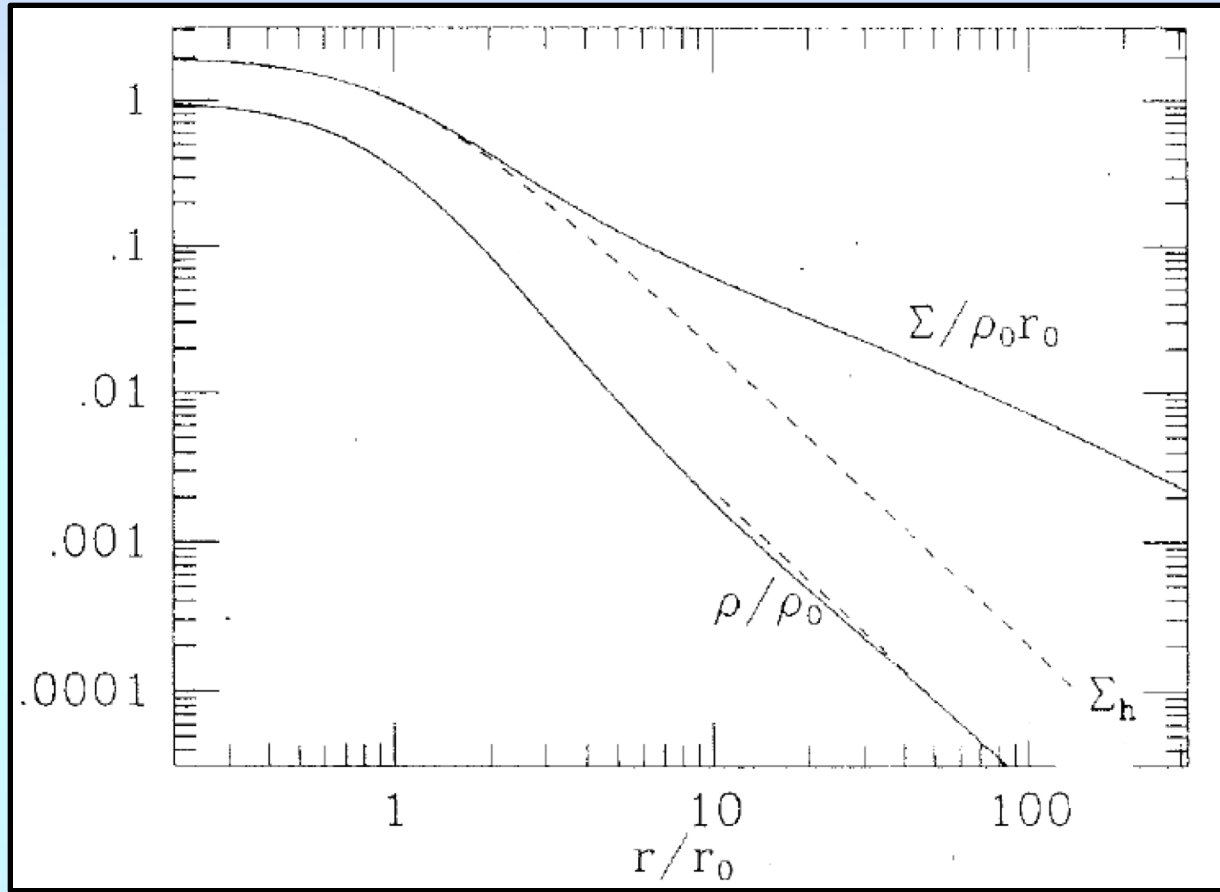
$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}$$

Of course, this solution has an infinite central density. To fix this, we can constrain the central density to be finite, and numerically solve for the density distribution.

Isothermal Spheres

The parameter r_0 is the “King radius”, or the “core radius”, where the projected mass density (Σ) has dropped by a factor of ~ 2 .

Because an isothermal sphere falls off as $1/r^2$ at large radii, the distribution implies a rotation speed that is independent of radius, and a total mass that is infinite.



$$r_0 = \left(\frac{3\langle v^2 \rangle}{4\pi G \rho_0} \right)^{1/2} = \left(\frac{9\sigma^2}{4\pi G \rho_0} \right)^{1/2} \quad \frac{\Sigma_0}{\rho_0 r_0} = 2.018$$

Isothermal Spheres

For $1/r^2$ distributions, $M(r) = \int_0^{r_t} 4\pi r^2 \rho(r) dr \propto \int_0^{r_t} K dr = K r_t$

where K is some constant. As $r_t \rightarrow \infty$, $M(r)$ becomes infinite. Also

$$\frac{GM(r)}{r^2} = \frac{v_c^2}{r} \Rightarrow v_c^2 \propto \frac{M(r)}{r} \propto \frac{K r}{r} = K$$

More specifically, if you keep track of the constants, $v_c = \sqrt{2} \sigma$.

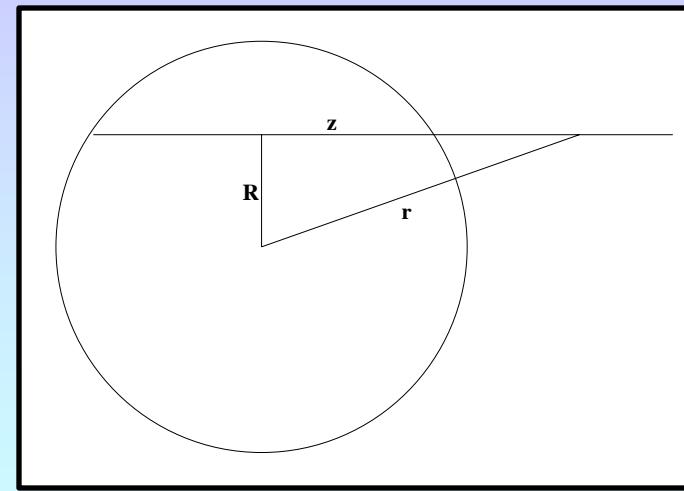
Note that the form of the isothermal sphere is numerical, but in the central regions ($r < 2 r_0$) a simple approximation (good to a couple of percent) is

$$\rho(r) = \frac{\rho_0}{\left\{1 + (r/r_0)^2\right\}^{3/2}} \quad \text{and} \quad \Sigma(R) = \frac{\Sigma_0}{1 + (R/R_0)^2}$$

Isothermal Spheres

The true density distribution and the projected density distribution are easily related.

$$\Sigma(R) = 2 \int_0^\infty \rho(z) dz = 2 \int_R^\infty \rho(r) \cdot \frac{r}{(r^2 - R^2)^{1/2}} dr$$



If we adopt $\rho(r) = \frac{\rho_0}{\left\{1 + (r/r_0)^2\right\}^{3/2}}$ then

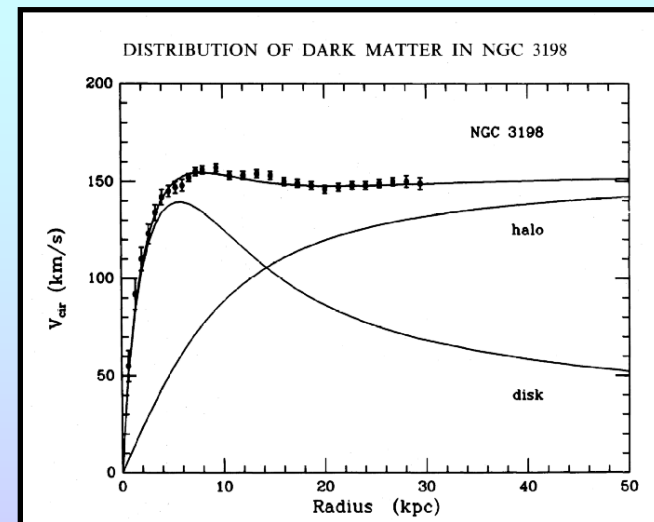
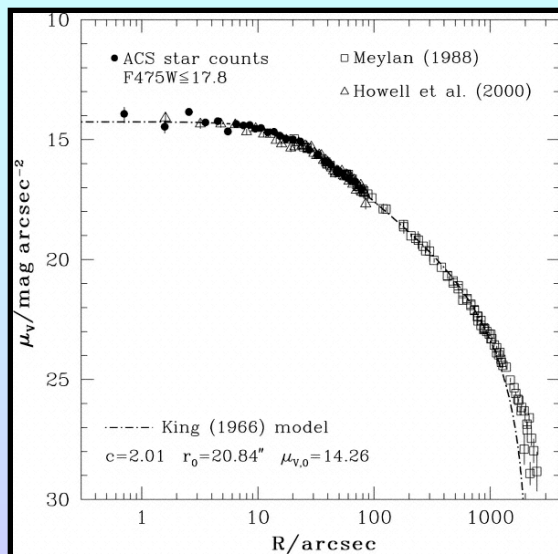
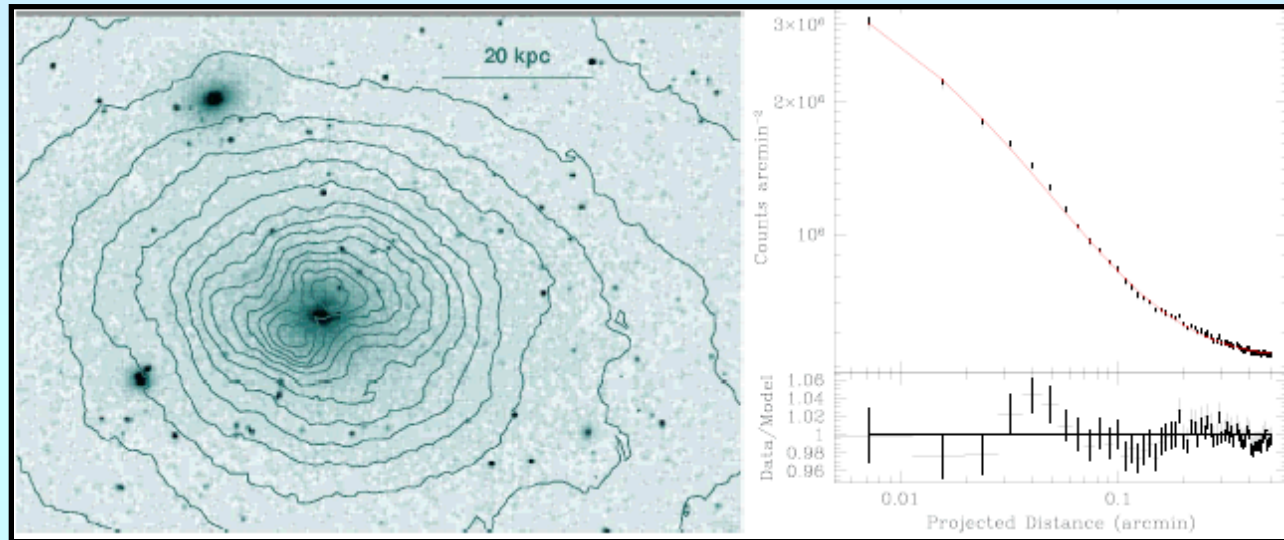
$$\Sigma(R) = 2\rho_0 \int_R^\infty \frac{r dr}{\left\{1 + (r/r_0)^2\right\}^{3/2} (r^2 - R^2)^{1/2}} = 2r_0^3 \rho_0 \int_R^\infty \frac{r dr}{(r_0^2 + r^2)^{3/2} (r^2 - R^2)^{1/2}}$$

Setting $\eta^2 = \frac{r^2 - R^2}{r_0^2 + R^2}$ then gives

$$\Sigma(R) = \frac{2\rho_0 r_0^3}{r_0^2 + R^2} \int_0^\infty \frac{d\eta}{(1 + \eta^2)^{3/2}} = \frac{2\rho_0 r_0}{1 + (R/r_0)^2}$$

Isothermal Spheres

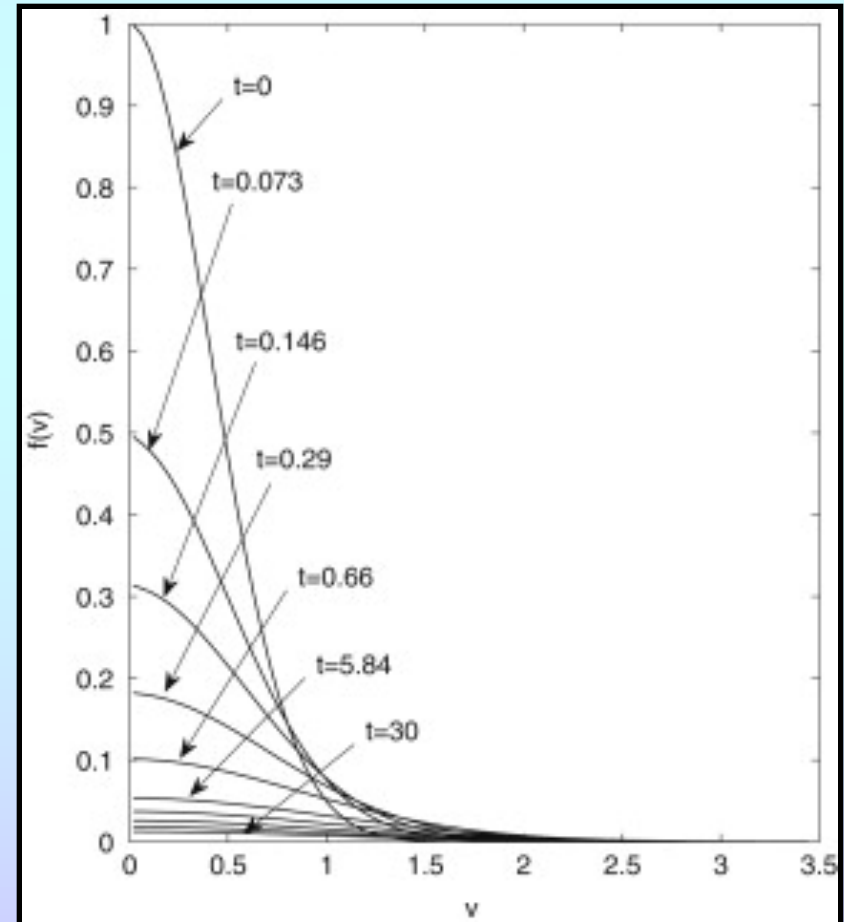
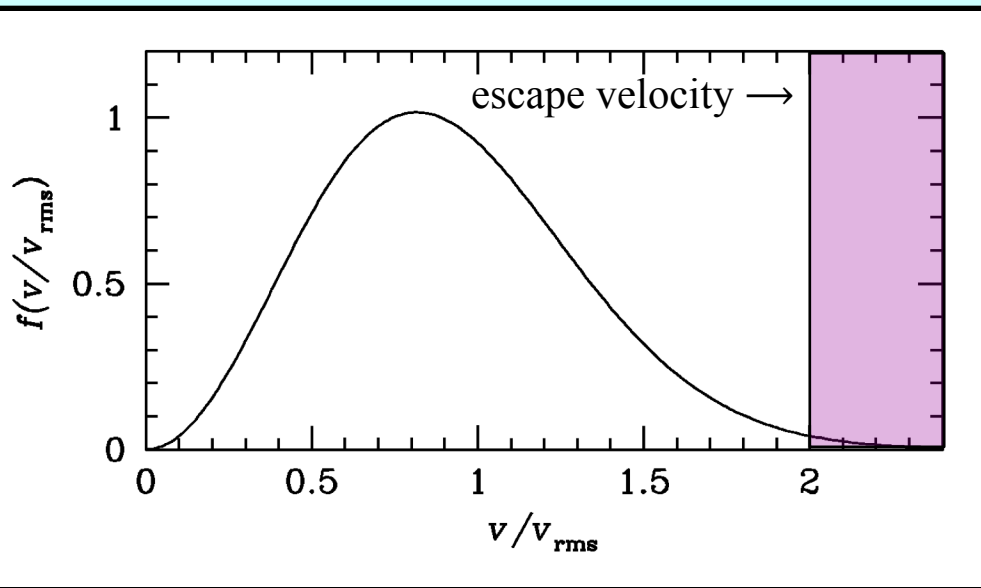
Note: isothermal spheres have many applications in astrophysics, including X-ray emission from galaxy clusters, dark matter distributions around galaxies, and, of course, the dynamics of star clusters.



Isothermal Spheres

Globular clusters are not exactly isothermal spheres because

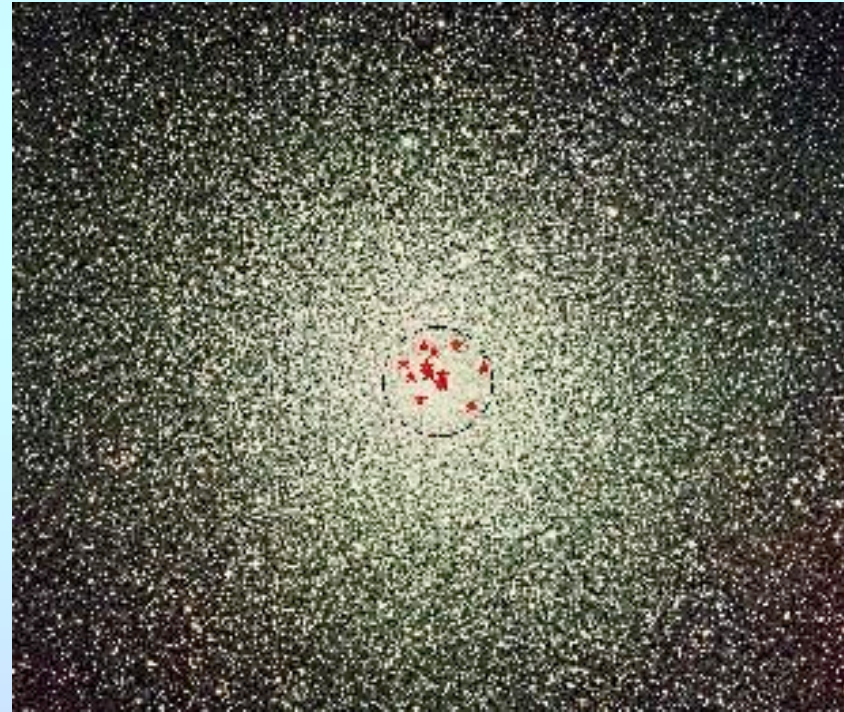
- Energy exchange continually populates the high-velocity tail of the Maxwellian distribution. These stars leave the cluster (i.e., evaporate), decreasing the cluster mass and potential, facilitating further evaporation.



Isothermal Spheres

Globular clusters are not exactly isothermal spheres because

- Energy exchange continually populates the high-velocity tail of the Maxwellian distribution. These stars leave the cluster (i.e., evaporate), decreasing the cluster mass and potential, facilitating further evaporation.
- Not all stars have the same mass: heavier particles sink to the cluster center, while less massive systems migrate outwards. This leads towards a “core collapse”.

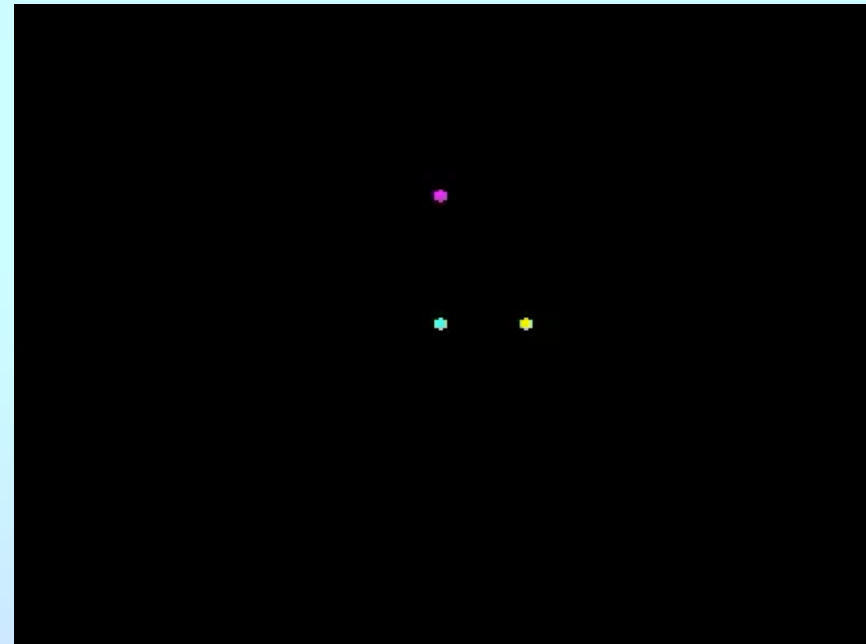


Positions of milli-second pulsars

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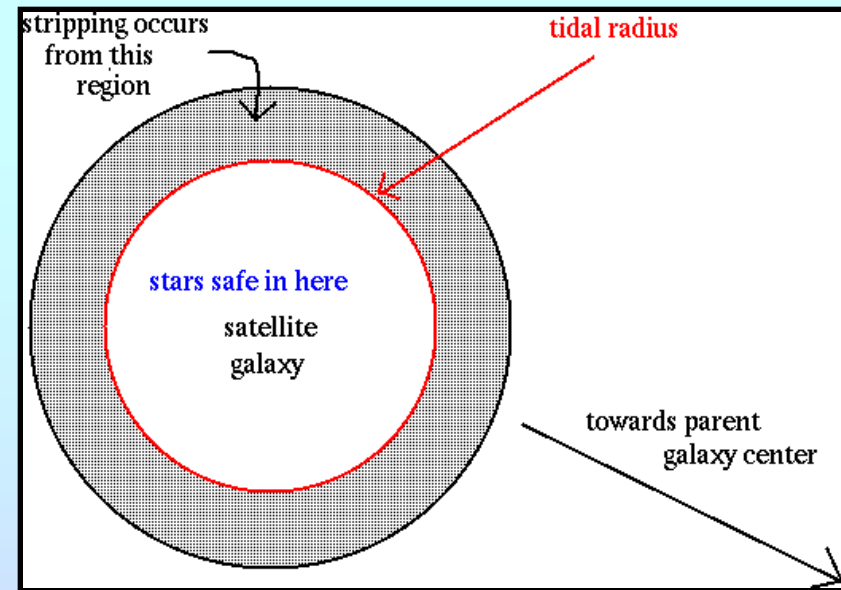
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- Binary stars take energy from the cluster, creating harder binaries and ejecting 3rd bodies. This energy prevents core collapse.



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- The Galactic tidal field will truncate the cluster.



Tidal Truncation

The cluster's motion through the Galaxy will have an effect on the system's structure. Consider a star at a distance r from a cluster's center. If the system has a galactocentric distance R , then the Galaxy will exert a force on the star

$$F_G = -\frac{GM_G}{R^2} \Rightarrow dF_G = \frac{2GM_G}{R^3} dr$$

Equating this to the cluster's own gravitational force yields

$$\frac{2GM_G}{R^3} r_t = \frac{GM_C}{r_t^2} \Rightarrow r_t = R \left(\frac{M_C}{2M_G} \right)^{1/3}$$

Actually, when one includes centripetal force and the fact that the cluster's orbit is likely elliptical with eccentricity, ε , and perigalacticon, R_p , the equation becomes

$$r_t = R_p \left\{ \frac{M_C}{M_G(3 + \varepsilon)} \right\}^{1/3}$$

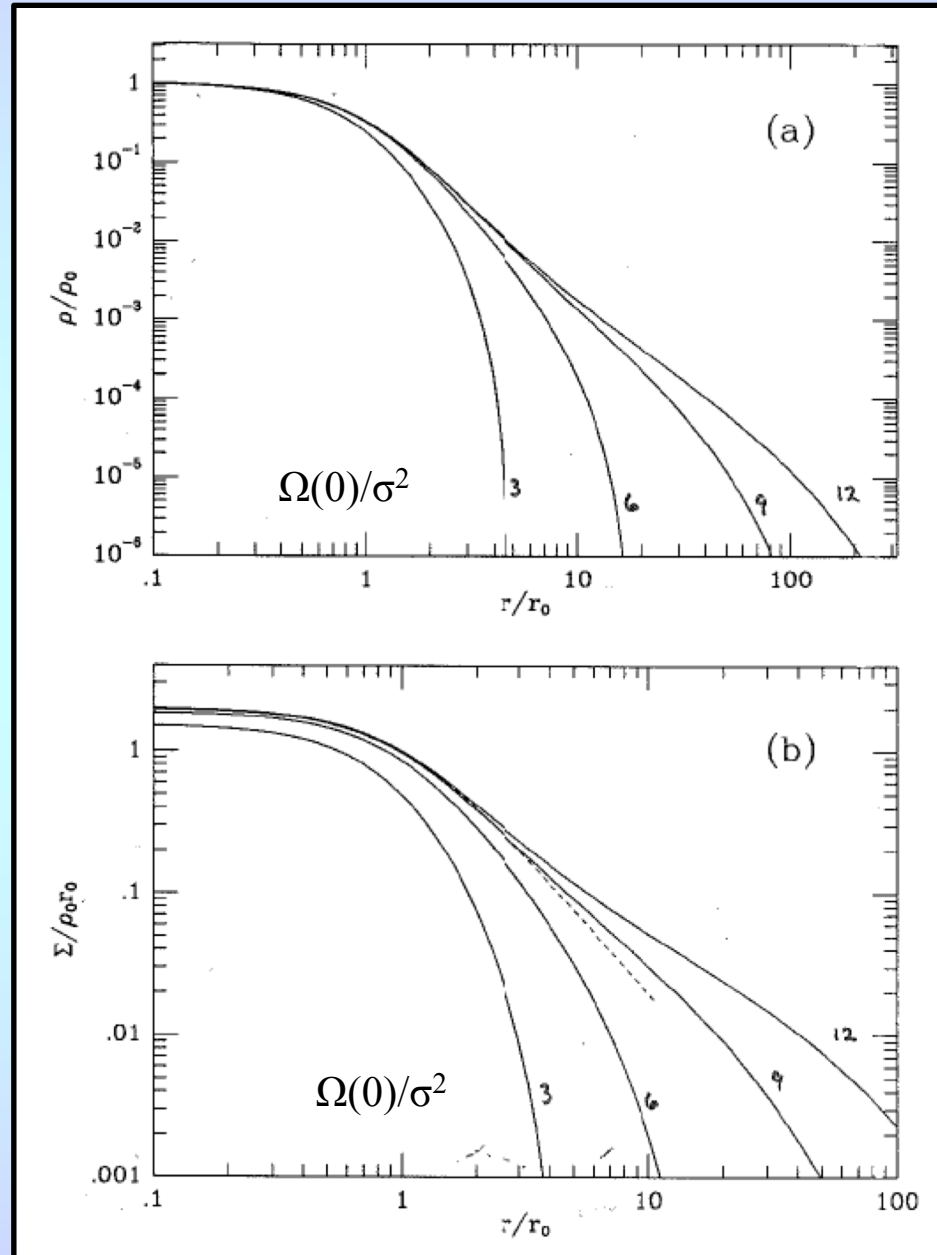
Lowered Isothermal Models

[King 1966, AJ, 71, 64]

To compensate for the effect of tides (and to fix the problem of infinite mass), King artificially truncated the isothermal distribution at low energies using a tidal energy. The result is a series of models defined by the ratio of the tidal radius to the core radius, $c = \log_{10}(r_t/r_0)$, or, alternatively, by the ratio of the central potential to the velocity dispersion, $\Omega(0)/\sigma^2$.

$$\rho(E)dE = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} \left\{ e^{-E/\sigma^2} - e^{-E_t/\sigma^2} \right\} dE$$

$$\text{where } E = \frac{1}{2}v^2 + \Omega$$



Lowered Isothermal Models

[King 1962, AJ, 67, 471]

Note that isothermal (and lowered isothermal) distributions are numerical. But a useful approximation for the projected surface brightness is

$$\Sigma(R) = \Sigma_0 \left\{ \frac{1}{\left[1 + (R/R_c)^2\right]^{1/2}} - \frac{1}{\left[1 + (R_t/R_c)^2\right]^{1/2}} \right\}^2$$